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Adaptive Coordinate Sampling for Stochastic Primal-Dual Optimization

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Abstract

We consider the regularized empirical risk minimization (ERM) of linear predictors, which arises in a variety of problems in machine learning and statistics. After reformulating the original ERM as a bilinear saddle-point problem, we can apply stochastic primal-dual methods to solve it. Sampling the primal or dual coordinates with a fixed non-uniform distribution is usually employed to accelerate the convergence of the algorithm, but such a strategy only exploits the global information of the objective function. To capture its local structures, we propose an adaptive importance sampling strategy that chooses the coordinates based on delicately-designed non-uniform and non-stationary distributions. In particular, we apply our sampling strategy to common stochastic primal-dual algorithms, including SPDC (Zhang and Xiao, 2017), DSPDC (Yu et al., 2015) and SPD1-VR (Tan et al., 2018). We show that our method has a linear convergence guarantee that is comparable to other methods, and we further show that a strictly sharper convergence rate can be obtained under certain conditions. Experimental results show that the proposed strategy significantly improves the convergence compared with common existing sampling methods.

Keywords: primal-dual methods; stochastic optimization algorithms; adaptive importance sampling; machine learning

1. Introduction

A wide range of problems in machine learning and statistics boil down to the following empirical risk minimization with linear predictor:

$$\min_{x\in\mathbb{R}^d}\left\{P(x)=\frac{1}{n}\sum_{i=1}^{n}\phi_i(a_i^\top x)+g(x)\right\}, \tag{1}$$

where $a_i\in\mathbb{R}^d$ is the i-th feature vector, $\phi_i:\mathbb{R}\to\mathbb{R}$ is the i-th convex closed loss function associated with the linear prediction $a_i^\top x$, and $g:\mathbb{R}^d\to\mathbb{R}$ is a convex regularization function for the predictor

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$x \in \mathbb{R}^d$. Problem (1) arises in many classification and regression tasks, where ϕ_i and g take different forms, such as

- logistic regression with sigmod loss:

$$\phi_i(z) = \log(1 + \exp(-b_i z)),\ b_i \in \{\pm 1\} \tag{2}$$

- support vector machine with smoothed hinge loss:

$$\phi_i(z) = \begin{cases} 0 & \text{if } b_i z \geq 1 \\ 1/2 - b_i z & \text{if } b_i z \leq 0 \\ (1/2)(1 - b_i z)^2 & \text{otherwise} \end{cases},\ b_i \in \{\pm 1\} \tag{3}$$

- linear/ridge regression with squared loss:

$$\phi_i(z) = \frac{1}{2}(z - b_i)^2,\ b_i \in \mathbb{R} \tag{4}$$

The commonly used regularizers include the ℓ_2 regularization $g(x) = (\lambda/2)\|x\|_2^2$ where $\lambda > 0$, and the $\ell_1 + \ell_2$ regularization $g(x) = \lambda_1\|x\|_1 + (\lambda_2/2)\|x\|_2^2$ where $\lambda_1, \lambda_2 > 0$.

Instead of directly solving the primal problem (1), it is often advantageous to tackle its equivalent primal-dual reformulation (Esser et al., 2010)

$$\min_{x \in \mathbb{R}^d} \max_{y \in \mathbb{R}^n} \left\{ F(x, y) = \frac{1}{n} y^\top A x - \frac{1}{n} \sum_{i=1}^{n} \phi_i^*(y_i) + g(x) \right\}, \tag{5}$$

where $A = [a_1^\top, \ldots, a_n^\top]^\top \in \mathbb{R}^{n \times d}$, $\phi_i^*(y_i) = \sup_{\nu \in \mathbb{R}}\{\nu y_i - \phi_i(\nu)\}$ is the convex conjugate function of ϕ_i. In this paper, we focus on the convex-concave saddle point problem (5). If ϕ_i is $(1/\gamma)$-smooth (i.e., ϕ_i has Lipschitz continuous gradient with constant $1/\gamma$), then its conjugate ϕ_i^* is γ-strongly convex (see Chapter E, Theorem 4.2.2 in Hiriart-Urruty and Lemaréchal (2012)). We consider the decomposable regularizer g, i.e.,

$$g(x) = \sum_{j=1}^{d} g_j(x_j), \tag{6}$$

where $g_j : \mathbb{R} \to \mathbb{R}$ is a univariate function of x_j. Obviously, both the ℓ_1 and ℓ_2 regularizations satisfy decomposability.

A favorable choice for solving problem (5) is the primal-dual hybrid gradient (PDHG) algorithm (Chambolle and Pock, 2011). In PDHG, the primal variable x and dual variable y are alternately updated by the proximal gradient method, and the sequence will finally converge to the saddle point of $F(x, y)$. Although PDHG is easy to implement, there is still computational bottleneck when the number of examples n is very large. To avoid evaluating full gradients in PDHG whose per-iteration complexity is $O(nd)$, a collection of stochastic primal-dual methods are proposed. Representative algorithms include SPDC (Zhang and Xiao, 2017), DSPDC (Yu et al., 2015) and SPD1-VR (Tan et al., 2018). SPDC introduces

randomness to the selection of dual coordinates, achieving significantly reduced $O(d)$ per-iteration computational cost. DSPDC and SPD1-VR are doubly stochastic algorithms where both the primal and dual coordinates are randomly selected. DSPDC and SPD1-VR have only $O(n+d)$ and $O(1)$ per-iteration complexity respectively, while SPD1-VR requires computation of full gradients once every fixed number of steps.

Although researchers have studied non-stationary sampling probabilities for stochastic gradient methods (Needell et al., 2014; Zhao and Zhang, 2015; Csiba and Richtárik, 2018; Zhou et al., 2018; Horváth and Richtarik, 2019; Qian et al., 2019), and random coordinate methods (Shalev-Shwartz and Tewari, 2011; Nesterov, 2012; Shalev-Shwartz and Zhang, 2013; Richtárik and Takáč, 2014; Lu and Xiao, 2015; Richtárik and Takáč, 2016; Shalev-Shwartz, 2016; Zhang and Gu, 2016) to solve the primal problem (1), there has not been work that equips stochastic primal-dual methods with adaptive sampling strategies. Taking SPDC as an example, the stationary sampling probability of the dual coordinate y_k is either $1/n$, which uses no extra information, or $(1-\delta)(1/n) + \delta(\|a_k\|_2 / \sum_{i=1}^n \|a_i\|_2)$, which only uses the global property of objective function. Hence, we are motivated to design varying sampling distributions that can exploit more local information of the objective function, which may further improve the convergence.

In this paper, we propose an adaptive coordinate sampling strategy, which dynamically adjusts the sampling distribution based on the first-order information collected in the past iterations. The sampling probability of each coordinate is weighted by the so-called gradient map, which is very lightweight in terms of both computation and storage. The probability update and adaptive sampling can be efficiently implemented using a binary tree data structure. Specifically, we propose SPDC-AIS, DSPDC-AIS and SPD1-VR-AIS, which are respectively SPDC, DSPDC and SPD1-VR incorporated with our adaptive sampling strategy. For SPDC-AIS, only dual coordinates are sampled based on some adaptive probabilities, while for DSPDC-AIS and SPD1-VR-AIS, both the primal and dual coordinates are sampled adaptively. We prove that the stochastic primal-dual algorithm based on our adaptive sampling converges linearly for strongly convex objective function. In addition, we theoretically show that the importance sampling can indeed contribute to faster convergence provided proper assumptions. Numerical evaluations on the widely used support vector machine (SVM) show that notably sharper convergence can be achieved compared with stochastic primal-dual methods with uniform sampling and traditional non-uniform sampling methods.

2. SINGLY ADAPTIVE SAMPLING

In this section, we introduce our adaptive sampling rule applied to SPDC, where only the dual coordinates are randomly chosen.

SPDC is basically an stochastic extension of PDHG (Chambolle and Pock, 2011) to approach the saddle point of $F(x,y)$ of problem (5). They both alternately maximize F with respect to y and minimize F with respect to x. If we rewrite problem (5) as

$$\min_{x\in\mathbb{R}^d} \max_{y\in\mathbb{R}^n} \left\{ F(x,y) = \frac{1}{n}\sum_{i=1}^n \left(\langle a_i, x\rangle y_i - \phi_i^*(y_i)\right) + g(x) \right\},$$

we can observe that for a fixed x, $F(x,y)$ is decomposable in terms of the dual coordinates y_i's. In the

t-th iteration of SPDC, for fixed $\bar{x}^t$, we first uniformly choose an index $i_t \in \{1, \ldots, n\}$ and then perform a proximal gradient ascent step on $\langle a_{i_t}, x\rangle\, y_{i_t} - \phi^*_{i_t}(y_{i_t})$. That is,

$$y_{i_t}^{t+1} = \mathrm{prox}_{\sigma\phi^*_{i_t}}(y_{i_t}^t + \sigma\left\langle a_{i_t}, \bar{x}^t\right\rangle) = \arg\max_{u\in\mathbb{R}}\left\{\left\langle a_{i_t}, \bar{x}^t\right\rangle u - \phi^*_{i_t}(u) - \frac{1}{2\sigma}(u - y_{i_t}^t)^2\right\}, \tag{7}$$

where σ is the dual step size. Subsequently, the whole primal vector is updated for fixed y^{t+1}:

$$x^{t+1} = \arg\min_{x\in\mathbb{R}^d}\left\{\left\langle s^t + (y_{i_t}^{t+1} - y_{i_t}^t)a_{i_t}, x\right\rangle + g(x) + \frac{1}{2\tau}\|x - x^t\|_2^2\right\}, \tag{8}$$

where τ is the primal step size and $s^t = (1/n)\sum_{i=1}^n y_i^t a_i \in \mathbb{R}^d$ can be pre-computed and stored. An extrapolation step is required to facilitate the convergence:

$$\bar{x}^{t+1} = x^{t+1} + \theta(x^{t+1} - x^t). \tag{9}$$

Steps (7)-(9) yield $O(d)$ overall computational cost, which is much lower than the $O(nd)$ per-iteration cost of PDHG.

2.1. SPDC-AIS

To make the algorithm more adaptive to the local structures of $F(x, y)$, we are motivated to design an adaptive coordinate sampling rule for SPDC. Rewrite equation (7) as

$$y_{i_t}^{t+1} = y_{i_t}^t + \sigma\mathcal{G}_\sigma(y_{i_t}^t), \tag{10}$$

where

$$\mathcal{G}_\sigma(y_{i_t}^t) = \frac{1}{\sigma}\left[\mathrm{prox}_{\sigma\phi^*_{i_t}}\left(y_{i_t}^t - \sigma\left\langle a_{i_t}, x^t\right\rangle\right) - y_{i_t}^t\right] \tag{11}$$

denotes the gradient map. By the first-order optimality condition of proximal mapping, $|\mathcal{G}_\sigma(y_i)| = 0$ if and only if y_i maximizes $\langle a_i, x\rangle\, y_i - \phi_i^*(y_i)$. Intuitively speaking, the larger the value of $|\mathcal{G}_\sigma(y_{i_t}^t)|$, the more often we wish to sample the i_t-th dual coordinate. Motivated by this, in the t-th iteration, we can sample the dual index $i \in \{1, \ldots, n\}$ with probability proportional to $|\mathcal{G}_\sigma(y_i^t)|^\kappa$, where κ usually takes non-negative values like 0, 0.5 and 1 (Allen-Zhu et al., 2016) (Nesterov, 2012). However, $|\mathcal{G}_\sigma(y_i^t)|^\kappa$ can be arbitrarily small, leading to little chance of sampling the i-th coordinate. Hence, a sampling distribution that is mixed with the uniform distribution is more reasonable. For instance, the probability of sampling i-th dual coordinate can be

$$p_i^t = (1 - \delta_t)\frac{1}{n} + \delta_t\frac{|\mathcal{G}_\sigma(y_i^t)|^\kappa}{\sum_{k=1}^n |\mathcal{G}_\sigma(y_k^t)|^\kappa},\ \forall i \in \{1, \ldots, n\}, \tag{12}$$

where $\delta \in (0, 1]$ is the parameter used to balance these two distribution. Hence, p_i^t is lower bounded, i.e., $p_i^t \geq (1 - \delta_t)/n$. Nevertheless, (12) requires the evaluation of $\mathcal{G}_\sigma(y_i^t)$ for all $i \in \{1, \ldots, n\}$, which takes

$O(nd)$ computational cost at every iteration and is thus not feasible. In our proposed method SPDC-AIS (see Algorithm 1 for details), we overcome this defect by replacing each $\mathcal{G}_\sigma(y_i^t)$ with $\mathcal{G}_\sigma(y_i^{[i]})$, where $[i]$ denotes the most recent iteration at which index i is picked. We need to store and maintain a vector $\pi = [\pi_1, \ldots, \pi_n]$, where $\pi_i = \mathcal{G}_\sigma(y_i^{[i]})$ for $i \in \{1, \ldots, n\}$. In other words, we use the historical gradient maps that are evaluated at different iterates to approximate the sampling probabilities (12), in exchange for significantly lower per-iteration computational cost.

Algorithm 1 SPDC-AIS

1: **Input:** primal step size $\tau > 0$, dual step size $\sigma > 0$, number of iterations T, initial points x^0 and y^0, parameters $\delta_t \in [\underline{\delta}, \overline{\delta}]$, $\kappa, \theta > 0$.
2: **Initialize:** $\bar{x}^0 = x^0$, $s^0 = (1/n)\sum_{k=1}^n y_k^0 a_k$, $\pi_i = 1$ for all $i \in \{1, \ldots, n\}$
3: **for** $t = 0, 1, 2, \ldots, T-1$ **do**
4: Update probability distribution p^t, where

$$p_i^t = (1 - \delta_t)\frac{1}{n} + \delta_t \frac{|\pi_i|^\kappa}{\sum_{k=1}^n |\pi_k|^\kappa}, \ \forall i \in \{1, \ldots, n\} \tag{13}$$

5: Randomly pick $i_t \in \{1, 2, \ldots, n\}$ according to the distribution p^t
6: Perform updates:

$$\begin{aligned}
y_{i_t}^{t+1} &= \arg\max_{\beta \in \mathbb{R}} \left\{ \left\langle a_i, \bar{x}^t \right\rangle \beta - \phi_i^*(\beta) - \frac{np_{i_t}^t}{2\sigma}(\beta - y_i^t)^2 \right\} \\
y_i^{t+1} &= y_i^t \text{ for all } i \neq i_t \\
\pi_{i_t} &= \frac{np_{i_t}^t}{\sigma}\left(y_{i_t}^{t+1} - y_{i_t}^t\right) \\
x^{t+1} &= \arg\min_{x \in \mathbb{R}^d} \left\{ \left\langle s^t + \frac{y_{i_t}^{t+1} - y_{i_t}^t}{np_{i_t}^t} a_{i_t}, x \right\rangle + g(x) + \frac{\|x - x^t\|_2^2}{2\tau} \right\} \\
s^{t+1} &= s^t + \frac{1}{n}(y_{i_t}^{t+1} - y_{i_t}^t)a_{i_t} \\
\bar{x}^{t+1} &= x^{t+1} + \theta(x^{t+1} - x^t)
\end{aligned}$$

7: **end for**
8: **Output:** x^T and y^T

2.2. Implementation of the Sampling

For SPDC-AIS, we need to choose indices according to a varying non-uniform distribution. In other words, we need to generate a random integer that follows a different distribution p in every iteration. To achieve this in a computationally efficient way, we resort to a binary tree data structure.

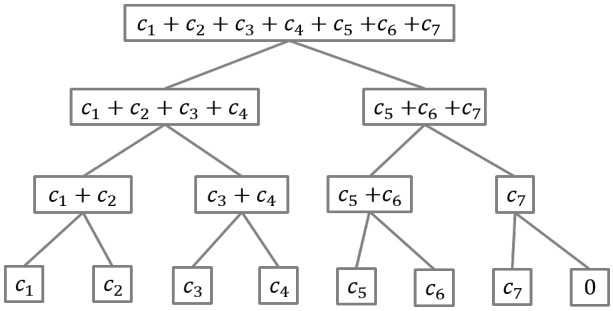


Fig. 1: Example of Binary Tree $\mathcal{T}_7$ (Each $c_i = |\pi_i|^\kappa$)

Suppose that there are n data examples. Let $c_i = |\pi_i|^\kappa$ be the i-th leaf of $\mathcal{T}_n$. By properly adding "empty" nodes, we can group the nodes at each level in pairs. Fig. 1 illustrates an example of such a binary tree with $n = 7$. Note that we need to add one more leaf at the bottom level, for grouping it with c_7. According to Algorithm 1, some π_i is changed in each iteration. Hence, all nodes related to c_i in $\mathcal{T}_n$ should be updated. The update of $\mathcal{T}_n$ can be done in a bottom-up way. It is known that the height of $\mathcal{T}_n$ is $\lceil \log n \rceil$. Thus, by updating the tree in a bottom-up approach, the computational cost is $O(\log n)$. $\mathcal{T}_n$ can contribute to generating a random integer following a given distribution. If we first generate a uniformly distributed random number $r \in [0, 1]$, we are supposed to find the index i such that $\sum_{k=1}^{i-1} p_k^t < r < \sum_{k=1}^{i} p_k^t$, where for $i \in \{1, \ldots, n\}$,

$$\sum_{k=1}^{i} p_k^t = \frac{(1-\delta_t)i}{n} + \delta_t \frac{\sum_{k=1}^{i} |\pi_k|^\kappa}{\sum_{l=1}^{n} |\pi_l|^\kappa}. \tag{14}$$

Since all the partial sums in the right-hand side of (14) are stored in $\mathcal{T}_n$, we can quickly find the desired i by visiting $\mathcal{T}_n$ in a top-down fashion. Obviously, searching for i also takes $O(\log n)$ time.

To conclude, additional $O(\log n)$ per-iteration cost is needed to implement the adaptive sampling. Since updating the primal and dual variables already requires $O(d)$ per-iteration cost, our adaptive coordinate sampling method does not increase the order of computational complexity.

3. DOUBLY ADAPTIVE SAMPLING

The adaptive coordinate sampling strategy can be extended to doubly stochastic algorithms DSPDC (Yu et al., 2015) and SPD1-VR (Tan et al., 2018).

Algorithm 2 DSPDC-AIS

1: **Input:** primal step size $\tau > 0$, dual step size $\sigma > 0$, number of iterations T, initial points x^0 and y^0, parameters $\delta_t \in [\underline{\delta}, \bar{\delta}]$, $\kappa, \theta > 0$.
2: **Initialize:** $\bar{x}^0 = x^0$, $s^0 = (1/n)\sum_{k=1}^n y_k^0 a_k$, $\pi_i = \psi_j = 1$ for all $i \in \{1, \ldots, n\}$ and $j \in \{1, \ldots, d\}$
3: **for** $t = 0, 1, 2, \ldots, T-1$ **do**
4: Update probability distribution p^t and q^t, where

$$p_i^t = (1 - \delta_t)\frac{1}{n} + \delta_t \frac{|\pi_i|^\kappa}{\sum_{k=1}^n |\pi_k|^\kappa}, \ \forall i \in \{1, \ldots, n\} \tag{15}$$

$$q_j^t = (1 - \delta_t)\frac{1}{d} + \delta_t \frac{|\psi_j|^\kappa}{\sum_{k=1}^d |\psi_k|^\kappa}, \ \forall j \in \{1, \ldots, d\} \tag{16}$$

5: Randomly pick $i_t \in \{1, 2, \ldots, n\}$ and $j_t \in \{1, 2, \ldots, d\}$ according to the distribution p^t and q^t, respectively
6: Perform updates:

$$\begin{aligned}
y_{i_t}^{t+1} &= \arg\max_{\beta \in \mathbb{R}} \left\{ \left\langle a_i, \bar{x}^t \right\rangle \beta - \phi_i^*(\beta) - \frac{np_{i_t}^t}{2\sigma}(\beta - y_{i_t}^t)^2 \right\} \\
y_i^{t+1} &= y_i^t \text{ for all } i \neq i_t \\
\pi_{i_t} &= \frac{np_{i_t}^t}{2\sigma}(y_{i_t}^{t+1} - y_{i_t}^t) \\
\bar{y}^{t+1} &= y^{t+1} + n(y^{t+1} - y^t) \\
x_{j_t}^{t+1} &= \arg\max_{\alpha \in \mathbb{R}} \left\{ \frac{1}{n} \left\langle A^j, \bar{y}^{t+1} \right\rangle \alpha - g_j(\alpha) - \frac{dq_{j_t}^t}{2\tau}(\alpha - x_j^t)^2 \right\} \\
x_j^{t+1} &= x_j^t \text{ for all } j \neq j_t \\
\psi_{j_t} &= \frac{dq_{j_t}^t}{2\tau}(x_{j_t}^{t+1} - x_{j_t}^t) \\
\bar{x}^{t+1} &= x^{t+1} + \theta(x^{t+1} - x^t)
\end{aligned}$$

7: **end for**
8: **Output:** x^T and y^T

3.1. DSPDC-AIS

As for DSPDC, instead of updating the whole vector x^{t+1}, we randomly sample $j_t \in \{1, 2, \ldots, d\}$ and update $x_{j_t}^{t+1}$ as

$$x_{j_t}^{t+1} = \arg\min_{\alpha \in \mathbb{R}} \left\{ \frac{1}{n} \left\langle A^{j_t}, \bar{y}^{t+1} \right\rangle \alpha + g_{j_t}(\alpha) + \frac{1}{2\tau}(\alpha - x_{j_t}^t)^2 \right\},$$

where A^j denotes j-th column of A and τ is the primal step size. The importance sampling strategy for primal variable x is similar to ones for dual variable y. Firstly, we define the gradient mapping for primal variables as

$$\mathcal{G}_\tau(x_j^t) = \left[x_j^t - \text{prox}_{\tau g_j}\left(x_j^t - \frac{\tau}{n}\left\langle A^j, y^t\right\rangle\right)\right]/\tau.$$

Then, the sampling distribution is also mixed with the uniform distribution, which is given by

$$q_j^t = (1-\delta_t)\frac{1}{d} + \delta_t \frac{|\mathcal{G}_\tau(x_j^t)|^\kappa}{\sum_{k=1}^d |\mathcal{G}_\tau(x_k^t)|^\kappa},\ \forall j \in \{1,\ldots,d\}. \tag{17}$$

To simplify the complexity, we use the historical gradient maps $\mathcal{G}_\tau(x_i^{[i]})$ evaluated in the most recent iteration to approximate $\mathcal{G}_\tau(x_i^t)$. Besides, we need to use another binary tree, whose i-th leaf is $|\psi_i| = |\mathcal{G}_\tau(x_i^{[i]})|$, to store the gradient information and achieve the non-uniform sampling. See Algorithm 2 for details.

3.2. SPD1-VR-AIS

SPD1-VR is a variant of DSPDC. Utilizing the decomposable structure of g as shown in (6), we can further rewrite problem (5) as

$$\min_{x\in\mathbb{R}^d}\max_{y\in\mathbb{R}^n}\left\{\frac{1}{n}\sum_{i=1}^n\sum_{j=1}^d\left(a_{ij}y_i x_j - \frac{1}{d}\phi_i^*(y_i) + g_j(x_j)\right)\right\}.$$

Observe that both primal and dual coordinates are decomposable by fixing the other one. We can randomly choose a primal index and a dual index in every iteration and achieve only $\mathcal{O}(1)$ per-iteration cost. Besides, variance reduction technique (Johnson and Zhang, 2013) is employed in SPD1-VR to accelerate the convergence, where full gradients with regard to both the primal and dual variables are re-computed once every fixed number of iterations.

Instead of picking the primal and dual coordinates uniformly, we propose SPD1-VR-AIS that incorporates SPD1-VR with our adaptive coordinate sampling strategy (see Algorithm 3). This is a double-loop algorithm and full gradients are computed periodically, which takes $O(nd)$ time every epoch. Instead of updating the probability distribution in every inner iteration, we do it right after computing the full gradients at the beginning of each epoch. The sampling probabilities are also defined by gradient maps same as (12) and (17). The computation of the proximal mappings takes only additional $O(n+d)$ time for each epoch. Since the epoch length is T, the sampling probability will be calibrated every T iterations. In other words, the sampling in the inner iterations uses historical information no more than T iterations ago. Moreover, non-uniform sampling is also performed at the beginning of each epoch, which guarantees that the per-iteration cost (including sampling and iteration) is still $O(1)$.

Algorithm 3 SPD1-VR-AIS

Input: primal step size $\tau > 0$, dual step size $\sigma > 0$, number of iterations T, initial points x^0 and y^0, parameters $\delta_t \in [\underline{\delta}, \bar{\delta}]$.
Initialize: $\tilde{x}^0 \in X$ and $\tilde{y}^0 \in Y$
for $k = 0, 1, 2, \ldots, K-1$ **do**
  Compute full gradients $G_x^k = (1/n)A^\top \tilde{y}^k$ and $G_y^k = (1/d)A\tilde{x}^k$
  Compute probability distribution p^k and q^k, where

$$p_i^k = (1-\delta_k)\frac{1}{n} + \delta_k \frac{|\mathcal{G}_\sigma(\tilde{y}_i^k)|^\kappa}{\sum_{l=1}^n |\mathcal{G}_\sigma(\tilde{y}_l^k)|^\kappa},\ \forall i \in \{1, \ldots, n\}$$

$$q_j^k = (1-\delta_k)\frac{1}{d} + \delta_k \frac{|\mathcal{G}_\tau(\tilde{x}_j^k)|^\kappa}{\sum_{l=1}^d |\mathcal{G}_\tau(\tilde{x}_l^k)|^\kappa},\ \forall j \in \{1, \ldots, d\}$$

  Independently sample $nd/\log(nd)$ primal indices following p^k, and $nd/\log(nd)$ dual indices following q^k, and store them in set I and J respectively
  Set $(x^0, y^0) = (\tilde{x}^0, \tilde{y}^0)$
  for $t = 0, 1, \ldots, T-1$ **do**
    Uniformly pick $i_t, i_t' \in I$ and $j_t, j_t' \in J$
    Perform updates:

$$\begin{aligned}
\bar{x}_{j_t}^t &= \text{prox}_{\tau g_{j_t}}\left(x_{j_t}^t - \tau\left(a_{i_t' j_t}(y_{i_t'}^t - \tilde{y}_{i_t'}^k) + G_{x,j_t}^k\right)\right)\\
\bar{y}_{i_t}^t &= \text{prox}_{(\sigma/d)\phi_{i_t}^*}\left(y_{i_t}^t - \sigma\left(a_{i_t j_t'}(x_{j_t'}^t - \tilde{x}_{j_t'}^k) + G_{y,i_t}^k\right)\right)\\
x_{j_t}^{t+1} &= \text{prox}_{\tau g_{j_t}}\left(x_{j_t}^t - \tau\left(a_{i_t j_t}(\bar{y}_{i_t}^t - \tilde{y}_{i_t}^k) + G_{x,j_t}^k\right)\right)\\
y_{i_t}^{t+1} &= \text{prox}_{(\sigma/d)\phi_{i_t}^*}\left(y_{i_t}^t - \sigma\left(a_{i_t j_t}(\bar{x}_{j_t}^t - \tilde{x}_{j_t}^k) + G_{y,i_t}^k\right)\right)\\
x_j^{t+1} &= x_j^t \text{ for all } j \neq j_t\\
y_i^{t+1} &= y_i^t \text{ for all } i \neq i_t
\end{aligned}$$

  end for
  Set $(\tilde{x}^{k+1}, \tilde{y}^{k+1}) = (x^T, y^T)$
end for
Output: $\tilde{x}^{k+1}$ and $\tilde{y}^{k+1}$

4. Convergence Analysis

In this section, we provide the theoretical analysis of the primal-dual adaptive coordinate sampling methods. We focus on the sampling methods based on SPDC.

SPDC-AIS shown in Algorithm 1 requires three control parameters τ, σ and θ, and its convergence is guaranteed provided that the parameters are properly specified and the sampling probabilities do not

vary drastically, which is stated in the following Theorem.

Theorem 1. *Let $\{x^t, y^t\}$ be the sequence generated by Algorithm 1. Suppose that each ϕ_i is convex and $(1/\gamma)$-smooth, g is λ-strongly convex. Denote $R := \max_i \|a_i\|_2$. The parameters τ, σ, θ in Algorithm 1 are chosen as*

$$\tau = \frac{1-\overline{\delta}}{2R}\sqrt{\frac{\gamma}{n\lambda}}, \quad \sigma = \frac{1-\overline{\delta}}{2R}\sqrt{\frac{n\lambda}{\gamma}}, \quad \theta = 1-\mu,$$

where $\mu = \min\left\{\frac{2\lambda\tau}{1+2\lambda\tau}, \frac{\gamma}{n/\sigma + n/(1-\overline{\delta})}\right\}$. Suppose that the following inequality holds for all $t \geq 1$:

$$\sum_{i=1}^{n}\left(\frac{1}{2\sigma} + \frac{\gamma(1-p_i^t)}{np_i^t}\right)(y_i^t - y_i^*)^2 \leq \theta \cdot \sum_{i=1}^{n}\left(\frac{1}{2\sigma} + \frac{\gamma}{np_i^{t-1}}\right)(y_i^t - y_i^*)^2, \tag{18}$$

then we have

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{2\tau}+\lambda\right)\|x^t - x^*\|_2^2 + \sum_{i=1}^{n}\left(\frac{1}{4\sigma}+\frac{\gamma}{n}\right)(y_i^t - y_i^*)^2\right] \\
&\leq \theta^t\left[\left(\frac{1}{2\tau}+\lambda\right)\|x^0 - x^*\|_2^2 + \sum_{i=1}^{n}\left(\frac{1}{2\sigma}+\gamma\right)(y_i^0 - y_i^*)^2\right],
\end{aligned}$$

where (x^*, y^*) is the saddle point.

Theorem 1 shows that the sequence $\{(x^t, y^t)\}$ will converge linearly to the unique saddle point in expectation, which matches the existing results in (Zhang and Xiao, 2017). Before we present the detailed proof of Theorem 1, we conduct simulations on three datasets (colon-cancer, a2a, gisette) to check the assumption (18). We obtain the almost optimal value y^* by running SPDC for sufficiently large number of iterations, then we can compute the term

$$\theta_t = \sum_{i=1}^{n}\left(\frac{1}{2\sigma} + \frac{\gamma(1-p_i^t)}{np_i^t}\right)(y_i^t - y_i^*)^2 \Bigg/ \sum_{i=1}^{n}\left(\frac{1}{2\sigma} + \frac{\gamma}{np_i^{t-1}}\right)(y_i^t - y_i^*)^2,$$

and compare θ_t with θ. The results are presented in Fig. 2, which shows that $\theta^t \leq \theta$ holds in most of the iterations. Therefore, we claim that the condition (18) holds in the sense of expectation.

To prove Theorem 1, we first state the following key lemma, which is a direct consequence of inequalities (62) and (63) in (Zhang and Xiao, 2017).

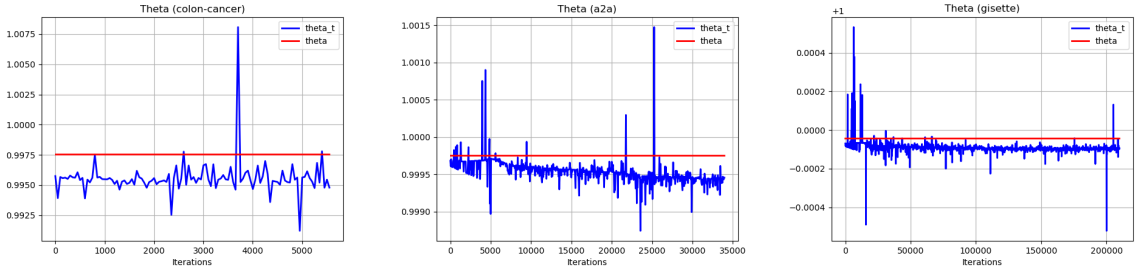


Fig. 2: Comparison of θ_t and θ for SPD1-AIS Algorithms

Lemma 1. *Let $\{(x^t, y^t)\}$ be the sequence generated by Algorithm 1. Then, we have*

$$\begin{aligned}
&\frac{\|x^t - x^*\|_2^2}{2\tau} + \sum_{i=1}^{n}\left(\frac{1}{2\sigma} + \frac{\gamma(1-p_i^t)}{np_i^t}\right)(y_i^t - y_i^*)^2 \geq \\
&\mathbb{E}\Bigg[\left(\frac{1}{2\tau} + \lambda\right)\|x^{t+1} - x^*\|_2^2 + \sum_{i=1}^{n}\left(\frac{1}{2\sigma} + \frac{\gamma}{np_{i_t}^t}\right)(y_{i_t}^{t+1} - y_{i_t}^*)^2 \\
&\quad + \frac{(y^{t+1} - y^*)^T A(x^{t+1} - x^t)}{n} - \frac{\theta(y^t - y^*)^T A(x^t - x^{t-1})}{n} \\
&\quad + \frac{\|x^{t+1} - x^t\|_2^2}{2\tau} + \frac{(y_{i_t}^{t+1} - y_{i_t}^t)^2}{2\sigma} \\
&\quad - \frac{1}{np_{i_t}^t}\|y_{i_t}^{t+1} - y_{i_t}^t\|_2\|a_{i_t}\|_2\left(\|x^{t+1} - x^t\|_2 + \theta\|x^t - x^{t-1}\|_2\right) \mid \mathcal{F}_t\Bigg].
\end{aligned} \tag{19}$$

Based on Lemma 1, we can obtain the following proposition.

Proposition 1. *Assume σ, τ are chosen such that $\sigma\tau \leq (1-\bar{\delta})^2/4R^2$. Then, we have*

$$\begin{aligned}
&\frac{\|x^t - x^*\|_2^2}{2\tau} + \sum_{i=1}^{n}\left(\frac{1}{2\sigma} + \frac{\gamma(1-p_i^t)}{np_i^t}\right)(y_i^t - y_i^*)^2 + \frac{\theta(y^t - y^*)^T A(x^t - x^{t-1})}{n} + \frac{\theta\|x^t - x^{t-1}\|_2^2}{4\tau} \\
&\geq \mathbb{E}\Bigg[(\frac{1}{2\tau} + \lambda)\|x^{t+1} - x^*\|_2^2 + \sum_{i=1}^{n}\left(\frac{1}{2\sigma} + \frac{\gamma}{np_i^t}\right)(y_i^{t+1} - y_i^*)^2 \\
&\qquad + \frac{(y^{t+1} - y^*)^T A(x^{t+1} - x^t)}{n} + \frac{\|x^{t+1} - x^t\|_2^2}{4\tau} \mid \mathcal{F}_t\Bigg].
\end{aligned}$$

Proof. We need to lower bound the last term on the right-hand side of inequality (19). Firstly, we have

$$\begin{aligned}\frac{1}{np_{i_t}^t}\|y_{i_t}^{t+1}-y_{i_t}^t\|_2\|a_{i_t}\|_2\|x^{t+1}-x^t\|_2 &\le \frac{\|x^{t+1}-x^t\|_2^2}{4\tau}+\frac{\tau}{(np_{i_t}^t)^2}\|a_{i_t}\|_2^2\|y_{i_t}^{t+1}-y_{i_t}^t\|_2^2\\ &\le \frac{\|x^{t+1}-x^t\|_2^2}{4\tau}+\frac{\tau R^2}{(1-\overline{\delta})^2}\|y_{i_t}^{t+1}-y_{i_t}^t\|_2^2\\ &\le \frac{\|x^{t+1}-x^t\|_2^2}{4\tau}+\frac{1}{4\sigma}\|y_{i_t}^{t+1}-y_{i_t}^t\|_2^2.\end{aligned}$$

The first inequality holds because of the Young's inequality; the second one holds since we know that $np_{i_t}^t \ge 1-\delta_t \ge 1-\overline{\delta}$ and $\|a_{i_t}\|_2 \le R$, while the last inequality holds because of the assumption that $\tau \le (1-\overline{\delta})^2/4R^2\sigma$. Similarly, we have

$$\frac{1}{np_{i_t}^t}\|y_{i_t}^{t+1}-y_t^{i_t}\|_2\|a_{i_t}\|_2\|x^t-x^{t-1}\|_2 \le \frac{\|x^t-x^{t-1}\|_2^2}{4\tau}+\frac{1}{4\sigma}\|y_{i_t}^{t+1}-y_{i_t}^t\|_2^2.$$

Thus, the last term on the right-hand side of inequality (19) can be lower bounded by

$$\begin{aligned}&\mathbb{E}\Bigg[\frac{\|x^{t+1}-x^t\|_2^2}{2\tau}+\frac{(y_{i_t}^{t+1}-y_{i_t}^t)^2}{2\sigma}\\ &\qquad -\frac{1}{np_{i_t}^t}\|y_{i_t}^{t+1}-y_{i_t}^t\|_2\|a_{i_t}\|_2\left(\|x^{t+1}-x^t\|_2+\theta\|x^t-x^{t-1}\|_2\right)\mid\mathcal{F}_t\Bigg]\\ \le\ &\mathbb{E}\left[\frac{\|x^{t+1}-x^t\|_2^2}{4\tau}-\theta\frac{\|x^t-x^{t-1}\|_2^2}{4\tau}\right].\end{aligned}$$

Combining the above inequality and (19) completes the proof. □

Now, we are ready to prove the Theorem 1.

Proof of Theorem 1. Define Δ^t ($t \ge 0$) as

$$\begin{aligned}\Delta^t=\mathbb{E}\Bigg[\left(\frac{1}{2\tau}+\lambda\right)\|x^t-x^*\|_2^2+\sum_{i=1}^n\left(\frac{1}{2\sigma}+\frac{\gamma}{np_i^{t-1}}\right)(y_i^t-y_i^*)^2\\ +\frac{(y^t-y^*)^TA(x^t-x^{t-1})}{n}+\frac{\|x^t-x^{t-1}\|_2^2}{4\tau}\Bigg].\end{aligned}$$

Firstly, based on these assignments of the parameters, we have

$$\frac{1/(2\tau)}{1/(2\tau)+\lambda} \le \theta.$$

Then, combining (18) and Proposition 1, we obtain the recursive relation $\Delta^{t+1} \leq \theta \cdot \Delta^t$. Thus,

$$\mathbb{E}\Bigg[\left(\frac{1}{2\tau}+\lambda\right)\|x^t-x^*\|_2^2+\sum_{i=1}^{n}\left(\frac{1}{2\sigma}+\frac{\gamma}{np_i^{t-1}}\right)(y_i^t-y_i^*)^2 \\ +\frac{(y^t-y^*)^TA(x^t-x^{t-1})}{n}+\frac{\|x^t-x^{t-1}\|_2^2}{4\tau}\Bigg]\leq\theta^t\Delta^0, \tag{20}$$

where

$$\Delta^0=\left(\frac{1}{2\tau}+\lambda\right)\|x^0-x^*\|_2^2+\sum_{i=1}^{n}\left(\frac{1}{2\sigma}+\gamma\right)(y_i^0-y_i^*)^2.$$

To bound the last two terms on the left-hand side of inequality (20), we note that

$$\begin{aligned}&\frac{(y^t-y^*)^TA(x^t-x^{t-1})}{n}\\ \geq&-\frac{\|x^t-x^{t-1}\|_2^2}{4\tau}-\frac{\tau\|y^t-y^*\|_2^2\|A\|_2^2}{n^2}\\ \geq&-\frac{\|x^t-x^{t-1}\|_2^2}{4\tau}-\frac{(1-\overline{\delta})^2\|y^t-y^*\|_2^2}{4n\sigma}\\ \geq&-\frac{\|x^t-x^{t-1}\|_2^2}{4\tau}-\frac{\|y^t-y^*\|_2^2}{4\sigma}.\end{aligned}$$

The first inequality holds because of the Young's inequality, and the second inequality follows from the assumption $\tau \leq (1-\overline{\delta})^2/4R^2\sigma$. Finally, we can simplify inequality (20) as

$$\mathbb{E}\left[\left(\frac{1}{2\tau}+\lambda\right)\|x^t-x^*\|_2^2+\sum_{i=1}^{n}\left(\frac{1}{4\sigma}+\frac{\gamma}{np_i^{t-1}}\right)(y_i^t-y_i^*)^2\right]\leq\theta^t\Delta^0. \tag{21}$$

Applying the fact that $p_i^{t-1} \leq 1$ to (21), the proof is completed. □

Theorem 1 demonstrates that SPDC-AIS has a comparable convergence rate with SPDC, while theoretically it does not demonstrate the advantages of importance sampling. In the following theorem, we show that importance sampling does enjoy a faster convergence rate. Before that, we need the following assumption.

Assumption 1. *There exists a constant $\rho > 0$, such that for any $y \in \mathbb{R}^n$ and $\sigma > 0$, we have*

$$\frac{\sum_{i=1}^n|\mathcal{G}_\sigma(y_i)|^3}{\sum_{i=1}^n|\mathcal{G}_\sigma(y_i)|}-\left(\frac{1}{n}\sum_{i=1}^{n}|\mathcal{G}_\sigma(y_i)|\right)^2\geq\frac{\rho}{n}\|y-y^*\|_2^2, \tag{22}$$

where $\mathcal{G}_\sigma(y_i)$ is the proximal mapping defined in (11).

According to the generalized mean inequality, we have

$$\frac{1}{n}\sum_{i=1}^{n}|\mathcal{G}_\sigma(y_i)|^3 \geq \left(\frac{1}{n}\sum_{i=1}^{n}|\mathcal{G}_\sigma(y_i)|\right)^3.$$

So the left-hand-side of (22) is always non-negative, while the equality holds only when all $\mathcal{G}_\sigma(y_i)$'s take the same value. Besides, we conduct simulations to approximate the exact value of ρ for three different datasets (colon-cancer, a2a, gisette). We firstly obtain the almost optimal value y^* by running SPDC for sufficiently large number of iterations, then we can compute the term at each iteration:

$$\rho_t = \left(\frac{\sum_{i=1}^{n}|\mathcal{G}_\sigma(y_i^t)|^3}{\sum_{i=1}^{n}|\mathcal{G}_\sigma(y_i^t)|} - \left(\frac{1}{n}\sum_{i=1}^{n}|\mathcal{G}_\sigma(y_i^t)|\right)^2\right) \Bigg/ \left(\frac{1}{n}\|y^t - y^*\|_2^2\right)$$

The results are shown in Fig. 3. Then, we choose the minimum value of ρ_t as a reasonable approximation

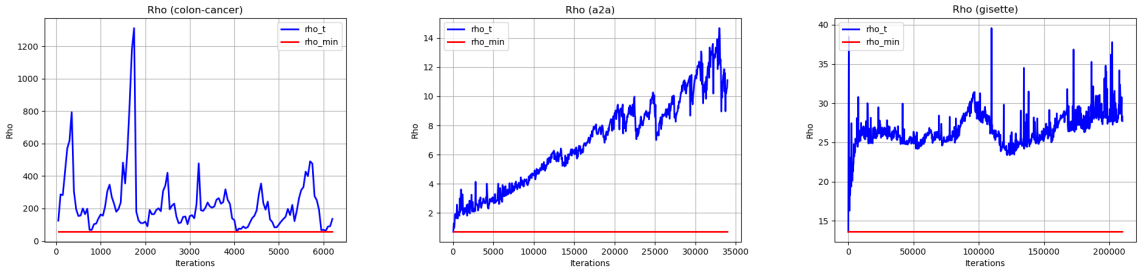


Fig. 3: The Value of ρ_t at Each Iteration for SPD1-AIS Algorithms

of ρ in Assumption 1. Table 1 reports the specific choices of ρ for three different datasets. As it is observed from Fig. 3 and Table 1, Assumption 1 always holds with some large enough constant $\rho > 0$, especially when y is close to y^*.

Table 1: Parameters for Different Datasets

	colon-cancer	a2a	gisette
ρ	56.6	0.7	13.6

Equipped with Assumption 1, we can establish an improved rate of convergence by adopting the adaptive probability distribution given in (12), which is stated in Theorem 2.

Theorem 2. *Suppose that each ϕ_i is convex, $(1/\gamma)$-smooth, g is λ-strongly convex and Assumption 1 holds. Besides, assume that*

$$p_i^t = (1-\delta_t)\frac{1}{n} + \delta_t \frac{|\mathcal{G}_\sigma(y_i^t)|}{\sum_{k=1}^{n}|\mathcal{G}_\sigma(y_k^t)|},\ \forall i \in \{1,\ldots,n\}.$$

Denote $R := \max_i \|a_i\|_2$. *If the parameters* τ, σ, θ *are chosen as*

$$\tau = \frac{1}{2R}\sqrt{\frac{\gamma}{n\lambda}}, \quad \sigma = \frac{1}{2R}\sqrt{\frac{n\lambda}{\gamma}}, \quad \theta = 1 - \tilde{\mu},$$

where $\tilde{\mu} = \min\left\{\frac{2\lambda\tau}{1+2\lambda\tau}, \frac{\gamma+\rho\sigma\underline{\delta}}{n/\sigma+n/(1-\overline{\delta})}\right\}$, *then we have*

$$\begin{aligned}
&\mathbb{E}\left[\left(\frac{1}{2\tau}+\lambda\right)\|x^t - x^*\|_2^2 + \sum_{i=1}^n \left(\frac{1}{4\sigma}+\frac{\gamma}{n}\right)(y_i^t - y_i^*)^2\right] \\
&\leq \theta^t \left[\left(\frac{1}{2\tau}+\lambda\right)\|x^0 - x^*\|_2^2 + \sum_{i=1}^n \left(\frac{1}{2\sigma}+\gamma\right)(y_i^0 - y_i^*)^2\right],
\end{aligned}$$

where (x^*, y^*) *is the saddle point.*

Proof. Firstly, we know that

$$\mathbb{E}\left[\frac{(y_{i_t}^{t+1} - y_{i_t}^t)^2}{2\sigma} \mid \mathcal{F}_t\right] = \sum_{i=1}^n \frac{\sigma}{2}|\mathcal{G}_\sigma(y_i^t)|^2 p_i^t = \frac{(1-\delta_t)\sigma}{2n}\sum_{i=1}^n |\mathcal{G}_\sigma(y_i^t)|^2 + \frac{\delta_t\sigma}{2}\frac{\sum_{i=1}^n |\mathcal{G}_\sigma(y_i^t)|^3}{\sum_{i=1}^n |\mathcal{G}_\sigma(y_i^t)|}. \tag{23}$$

By the definition of p_i^t and the fact that $(ax+by)(a/x+b/y) \geq (a+b)^2$ for all x, y, a, $b > 0$, we have

$$\frac{1}{p_i^t} \leq (1-\delta_t)n + \delta_t \frac{\sum_{k=1}^n |\mathcal{G}_\sigma(y_k^t)|}{|\mathcal{G}_\sigma(y_i^t)|}. \tag{24}$$

Thus, we lower bound the last term on the right-hand side of inequality (19)

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{np_{i_t}^t}\|y_{i_t}^{t+1} - y_{i_t}^t\|_2 \|a_{i_t}\|_2 \|x^{t+1} - x^t\|_2 \mid \mathcal{F}_t\right] \\
&\leq \mathbb{E}\left[\frac{\|x^{t+1} - x^t\|_2^2}{4\tau} + \frac{\tau}{(np_{i_t}^t)^2}\|a_{i_t}\|_2^2 \|y_{i_t}^{t+1} - y_{i_t}^t\|_2^2 \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\frac{\|x^{t+1} - x^t\|_2^2}{4\tau} \mid \mathcal{F}_t\right] + \sum_{i=1}^n \frac{\tau R^2}{n^2 p_i^t}\sigma^2 |\mathcal{G}_\sigma(y_i^t)|^2 \\
&\leq \mathbb{E}\left[\frac{\|x^{t+1} - x^t\|_2^2}{4\tau} \mid \mathcal{F}_t\right] + \frac{(1-\delta_t)\sigma}{4n}\sum_{i=1}^n |\mathcal{G}_\sigma(y_i^t)|^2 + \frac{\delta_t\sigma}{4n^2}\left(\sum_{i=1}^n |\mathcal{G}_\sigma(y_i^t)|\right)^2,
\end{aligned} \tag{25}$$

where the last inequality holds because of (24) the fact that $\tau\sigma = 1/4R^2$. Similarly with (25), we also

have

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{np_{i_t}^t}\|y_{i_t}^{t+1}-y_{i_t}^t\|_2\|a_{i_t}\|_2\|x^t-x^{t-1}\|_2 \mid \mathcal{F}_t\right] \\
\leq &\mathbb{E}\left[\frac{\|x^t-x^{t-1}\|_2^2}{4\tau} \mid \mathcal{F}_t\right]+\frac{(1-\delta_t)\sigma}{4n}\sum_{i=1}^n|\mathcal{G}_\sigma(y_i^t)|^2+\frac{\delta_t\sigma}{4n^2}\left(\sum_{i=1}^n|\mathcal{G}_\sigma(y_i^t)|\right)^2.
\end{aligned} \tag{26}$$

Combining (23), (25) and (26) together gives

$$\begin{aligned}
&\mathbb{E}\Bigg[\frac{\|x^{t+1}-x^t\|_2^2}{2\tau}+\frac{(y_{i_t}^{t+1}-y_{i_t}^t)^2}{2\sigma} \\
&\qquad -\frac{1}{np_{i_t}^t}\|y_{i_t}^{t+1}-y_{i_t}^t\|_2\|a_{i_t}\|_2(\|x^{t+1}-x^t\|_2+\theta\|x^t-x^{t-1}\|_2) \mid \mathcal{F}_t\Bigg] \\
\geq &\mathbb{E}\left[\frac{\|x^{t+1}-x^t\|_2^2}{4\tau}-\theta\frac{\|x^t-x^{t-1}\|_2^2}{4\tau} \mid \mathcal{F}_t\right]+\frac{\delta_t\sigma}{2}\left(\frac{\sum_{i=1}^n|\mathcal{G}_\sigma(y_i^t)|^3}{\sum_{i=1}^n|\mathcal{G}_\sigma(y_i^t)|}-(\frac{1}{n}\sum_{i=1}^n|\mathcal{G}_\sigma(y_i^t)|)^2\right) \\
\geq &\mathbb{E}\left[\frac{\|x^{t+1}-x^t\|_2^2}{4\tau}-\theta\frac{\|x^t-x^{t-1}\|_2^2}{4\tau} \mid \mathcal{F}_t\right]+\frac{\delta_t\sigma\rho}{2n}\|y^t-y^*\|_2^2,
\end{aligned} \tag{27}$$

where the last inequality results from Assumption 1. Further by Lemma 1, we obtain an improved version of Proposition 1:

$$\begin{aligned}
&\frac{\|x^t-x^*\|_2^2}{2\tau}+\sum_{i=1}^n\left(\frac{1}{2\sigma}-\frac{\delta_t\sigma\rho}{2}+\frac{\gamma(1-p_i^t)}{np_i^t}\right)(y_i^t-y_i^*)^2 \\
&\qquad +\frac{\theta(y^t-y^*)^TA(x^t-x^{t-1})}{n}+\theta\frac{\|x^t-x^{t-1}\|_2^2}{4\tau} \\
\geq &\mathbb{E}\Bigg[(\frac{1}{2\tau}+\lambda)\|x^{t+1}-x^*\|_2^2+\sum_{i=1}^n\left(\frac{1}{2\sigma}+\frac{\gamma}{np_i^t}\right)(y_i^{t+1}-y_i^*)^2 \\
&\qquad +\frac{(y^{t+1}-y^*)^TA(x^{t+1}-x^t)}{n}+\frac{\|x^{t+1}-x^t\|_2^2}{4\tau} \mid \mathcal{F}_t\Bigg].
\end{aligned}$$

The remaining part of the proof is just the same as the proof of Theorem 1. Consequently we have completed the proof of Theorem 2. □

Comparing Theorem 2 with Theorem 1, we can see that $\tilde{\mu} > \mu$, which actually indicates the benefit brought by the importance sampling. In other words, under proper assumptions, the theoretical convergence rate is improved from $1-\mu$ to $1-\tilde{\mu}$.

Remark 1. According to the Theorem 1 in (Zhang and Xiao, 2017), SPDC with the same parameters τ

and σ achieve the following convergence rate:

$$\mathbb{E}[\Delta^{(t)}] \leq \overline{\theta}^t \left(\Delta^{(0)} + \frac{\|y^{(0)} - y^\star\|_2^2}{4\sigma} \right) \quad \text{where} \quad \overline{\theta} = 1 - \left(n + 2R\sqrt{\frac{n}{\lambda\gamma}} \right)^{-1}.$$

To compare the convergence rate between SPDC and SPDC-AIS, we only need to compare $\overline{\theta}$ and θ given in Theorem 2. Firstly,

$$\begin{aligned}
\theta = 1 - \tilde{\mu} &= 1 - \min\left\{ \frac{2\lambda\tau}{1+2\lambda\tau}, \frac{\gamma + \rho\sigma\underline{\delta}}{n/\sigma + n/(1-\overline{\delta})} \right\} \\
&= \max\left\{ 1 - \left(1 + R\sqrt{\frac{n}{\lambda\gamma}}\right)^{-1}, 1 - \left(\frac{n}{(\gamma+\rho\sigma\underline{\delta})(1-\overline{\delta})} + \frac{\gamma}{\gamma+\rho\sigma\underline{\delta}} \cdot 2R\sqrt{\frac{n}{\lambda\gamma}} \right)^{-1} \right\}
\end{aligned}$$

Assume $\gamma \geq 1$ (which is true for all these three loss functions mentioned in Introduction), by choosing $\overline{\delta} = 1 - 1/(\gamma + \rho\sigma\underline{\delta}) \in (0,1)$, we have

$$\begin{aligned}
\theta &= \max\left\{ 1 - \left(1 + R\sqrt{\frac{n}{\lambda\gamma}}\right)^{-1}, 1 - \left(n + \frac{\gamma}{\gamma+\rho\sigma\underline{\delta}} \cdot 2R\sqrt{\frac{n}{\lambda\gamma}} \right)^{-1} \right\} \\
&\leq 1 - \left(n + \max\left\{ \frac{\gamma}{\gamma+\rho\sigma\underline{\delta}}, \frac{1}{2} \right\} \cdot 2R\sqrt{\frac{n}{\lambda\gamma}} \right)^{-1} \\
&< 1 - \left(n + 2R\sqrt{\frac{n}{\lambda\gamma}} \right)^{-1} \\
&= \overline{\theta},
\end{aligned}$$

where the first inequality is due to $\max\left\{ \frac{\gamma}{\gamma+\rho\sigma\underline{\delta}}, \frac{1}{2} \right\} < 1$. Thus, ρ depends how much the improvement of convergence rate is. Besides, if there is no importance sampling, i.e., $\underline{\delta} = \overline{\delta} = 0$, then we have $\theta = \overline{\theta}$, which means that SPDC-AIS reduces to the standard SPDC.

5. EXPERIMENTAL RESULTS

In this section, we present the experiments based on ℓ_2-regularized support vector machine (SVM) with smoothed hinge loss. Specifically, the objective function is $P(x) = \frac{1}{n}\sum_{i=1}^n \phi_i(a_i^\top x) + \frac{\lambda}{2}\|x\|_2^2$, where ϕ_i is defined in (3). For both the singly stochastic and doubly stochastic primal-dual frameworks, we compare our adaptive importance sampling (AIS) method with other sampling methods, i.e., stationary Lipschitz-based importance sampling (LIS) and uniform sampling (US). All the algorithms are tested based on real datasets a2a, w8a, gisette and colon-cancer, where the values of λ are set as $10^{-2}, 10^{-2}, 10^{-1}, 10^{0}$ respectively. The attributes of these datasets and values of λ chosen for each dataset are summarized in Table 2. The datasets basically cover three different types, where $n \gg d$, $n \approx d$ and $n \ll d$, respectively.

Table 2: Parameters for Different Datasets

	colon-cancer	gisette	a2a	w8a
n	62	6000	2265	49746
d	2000	5000	123	300
λ	10^0	10^{-1}	10^{-2}	10^{-2}

5.1. Experiments on SPDC-AIS

We compare the performance of three algorithms, which are respectively SPDC-AIS, SPDC-LIS and SPDC-US. For SPDC-AIS, we set $\kappa = 0.5$, $\underline{\delta} = 0.2$, $\overline{\delta} = 0.8$, and $\delta_t = \underline{\delta} + (\overline{\delta} - \underline{\delta})t/T$, where T is the maximum number of iterations we run. All the hyper-parameters τ, σ, θ are chosen by their theoretical values given in Theorem 1. For SPDC-LIS, we let the probability of sampling the i-th dual coordinate be $p_i^L = (1-\delta_t)\frac{1}{n} + \delta_t \frac{\|a_i\|_2}{\sum_{k=1}^n \|a_k\|_2}$, where $\|a_i\|_2$ is the Lipschitz constant of the component gradient $a_i \nabla \phi_i(a_i^\top x)$. As shown in Fig. 4, all three algorithms achieve linear convergence, while our proposed SPDC-AIS always exhibits notably sharper convergence rate than other two algorithms. As for SPDC-LIS, it outperforms uniform sampling only in the case of colon-cancer dataset where $n \ll d$. For other three datasets, SPDC-LIS is just comparable with SPDC-US.

We also conduct experiments on two high dimensional datasets, i.e., RCV1 and Covtype. Besides, as suggested by (Zhang and Xiao, 2017), we choose three small regularization weights $\lambda = 10^{-4}, 10^{-6}, 10^{-8}$ to guarantee good accuracy. The results are reported in Table 3, which shows that our proposed SPDC-AIS always exhibits notably sharper convergence rate than other two algorithms.

5.2. Experiments on DSPDC-AIS & SPD1-VR-AIS

Similar to SPDC-based algorithms, we also test the performance of doubly stochastic algorithms. For DSPDC-AIS and SPD1-VR-AIS, we also set $\kappa = 0.5$, and $\delta_k = \underline{\delta} + (\overline{\delta} - \underline{\delta})k/K$, where K is the maximum number of epochs we run. We adopt best-tuned fixed stepsizes for the three algorithms. For DSPDC-LIS and SPD1-VR-LIS, we let the probability of sampling the i-th dual coordinate and j-th primal coordinate be respectively $p_i^L = (1-\delta_k)\frac{1}{n} + \delta_k \frac{\|a_i\|_2}{\sum_{k=1}^n \|a_k\|_2}$ and $q_j^L = (1-\delta_k)\frac{1}{n} + \delta_k \frac{\|\alpha_j\|_2}{\sum_{l=1}^d \|\alpha_l\|_2}$.

As shown in Fig. 5 and Fig. 6, our adaptive sampling-based algorithms converge much faster than the other two sampling methods for all datasets, while LIS-based algorithms has similar overall performance with US-based algorithms. These empirical results on real datasets justify that our proposed adaptive importance sampling method noticeably accelerates the convergence of stochastic primal-dual algorithms.

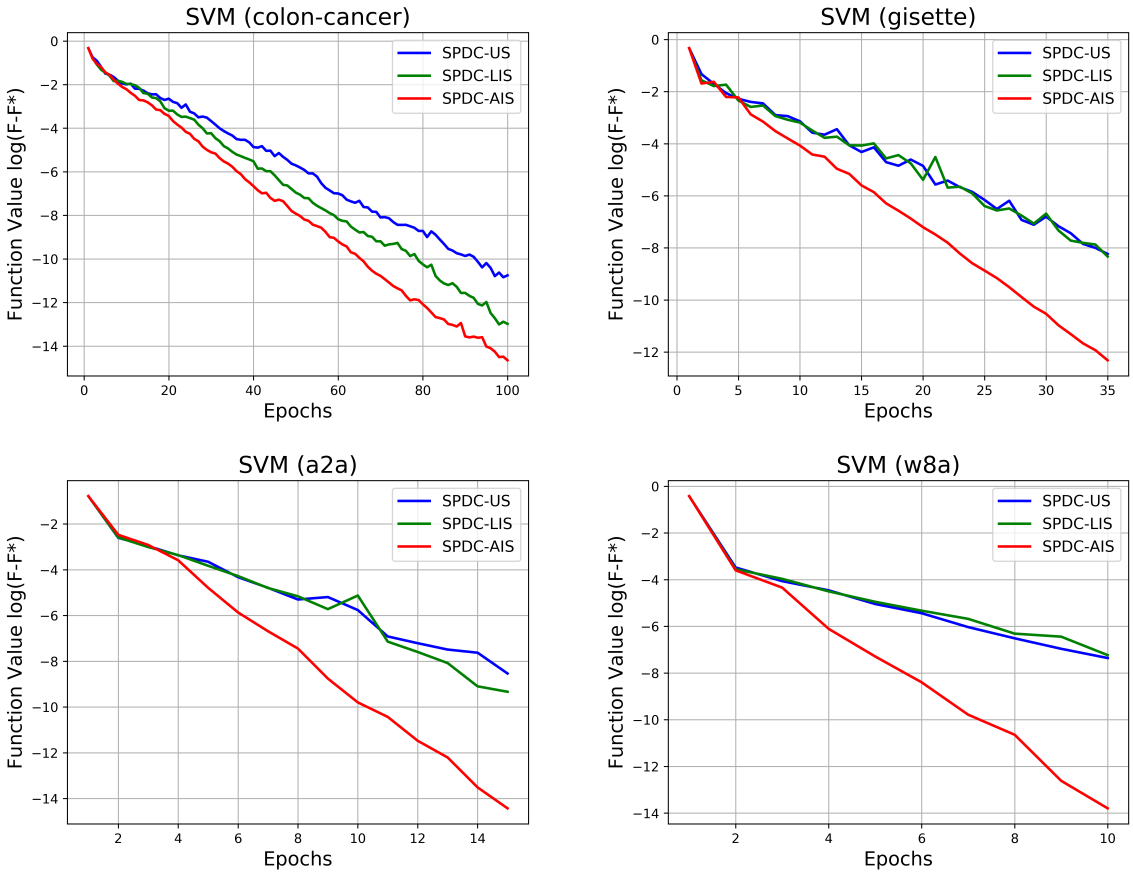


Fig. 4: Experimental Results for SPDC-based Algorithms

5.3. Results of Execution Time

For fair comparisons of the empirical results, we also provide the execution time of the experiments in section 5.1 and 5.2. All our experiments are conducted based on an Intel i5 processor with 3.1GHz main frequency. Table 2-4 presents the specific running time (in seconds) per epoch. As expected, non-uniform sampling methods is somewhat more time-consuming than uniform sampling, since non-uniform sampling requires $O(\log n)$ time per iteration to generate a random number. Besides, adaptive sampling takes a little more time than Lipschitz-based sampling, since adaptive sampling requires extra $O(\log n)$ to update the binary tree. For the same dataset, the execution time does not differ much from each other. Particularly, for dataset with relatively large dimension d, the extra execution time of our adaptive sampling is just marginal relative to the $O(d)$ time for updating the primal and dual variables. Together with results in Fig. 4-6, we conclude that at the cost slightly higher computational burden per epoch, our adaptive coordinate sampling methods significantly boost the stochastic primal-dual algorithms.

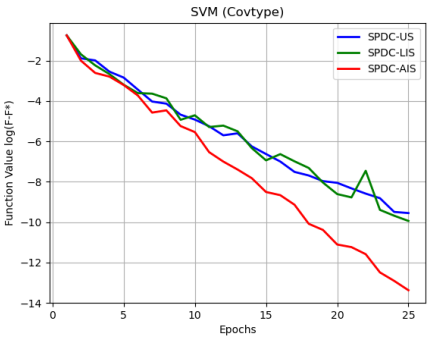
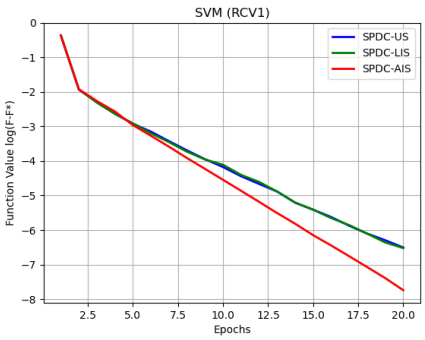
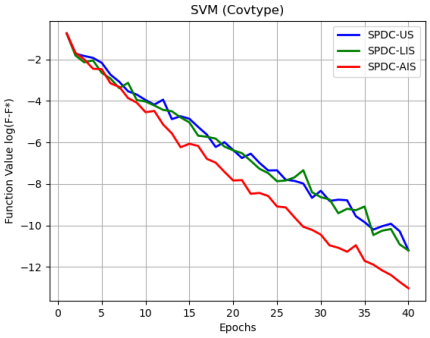
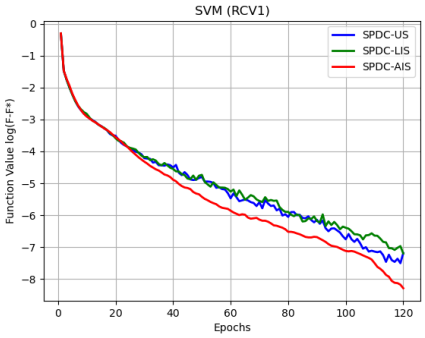
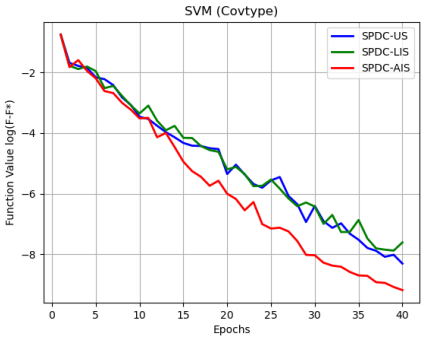
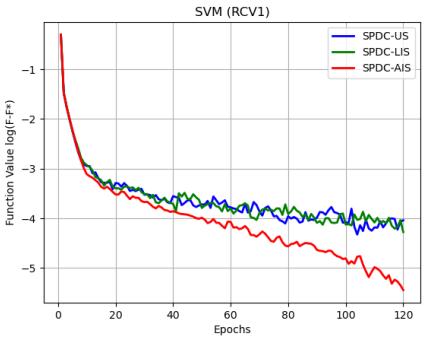
λ	Covtype	RCV1
10^{-4}		
10^{-6}		
10^{-8}		

Table 3: Experimental Results of SDPC-based Algorithm for Different Regularizer Weights

6. Conclusion

In this paper, we have investigated an adaptive importance sampling method for stochastic primal-dual optimization algorithms. The proposed method samples the primal and dual coordinates by adapting to

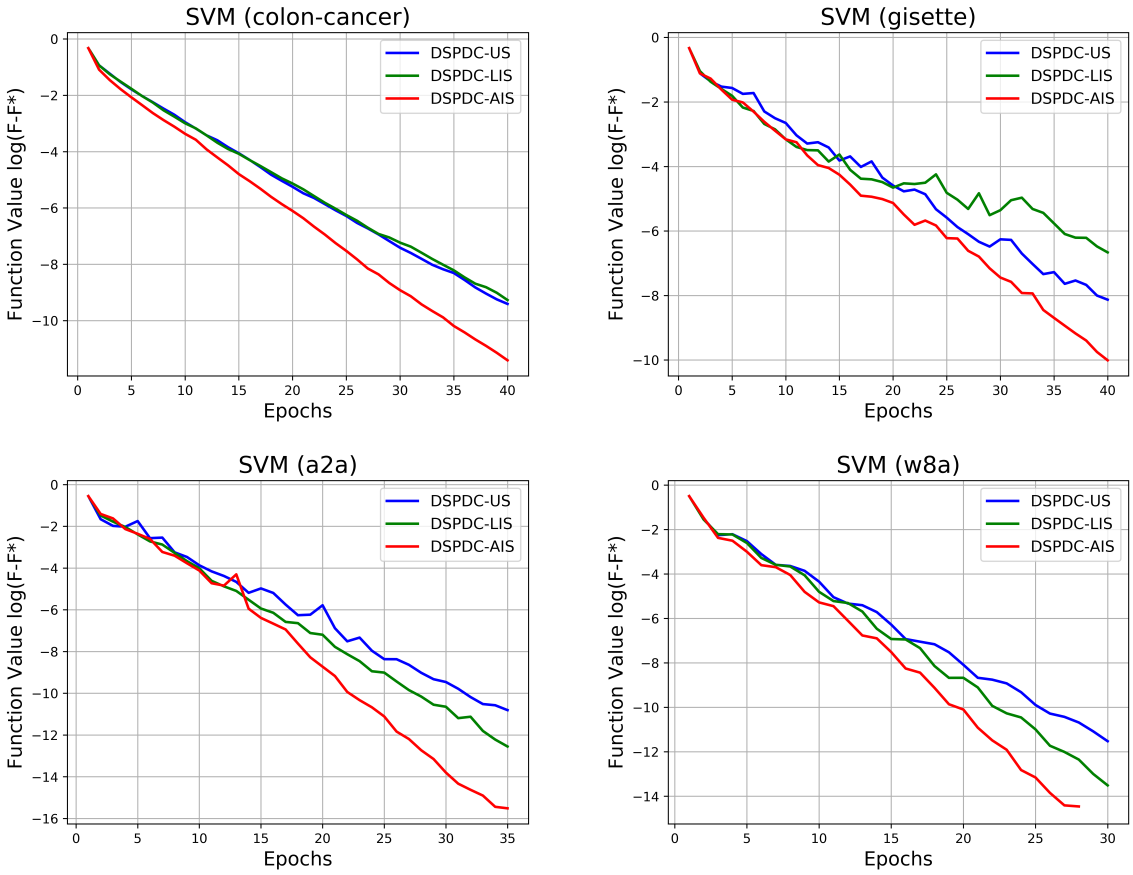


Fig. 5: Experimental Results for DSPDC-based Algorithms

Table 4: Execution Time of SPDC-based Algorithms

	colon-cancer	gisette	a2a	w8a
US	0.0042	3.01	0.0060	0.796
LIS	0.0043	3.09	0.0067	0.839
AIS	0.0049	3.34	0.008	0.895

the local structure of the objective function. We take advantage of a specific binary tree structure to implement computationally efficient sampling. We apply our sampling method to three common stochastic primal-dual algorithms, i.e., SPDC, DSPDC and SPD1-VR. Detailed theoretical analysis is provided to demonstrate the effectiveness of our methods. Experiments on real datasets verify that our adaptive coordinate sampling achieves significantly faster convergence than common stationary sampling methods.

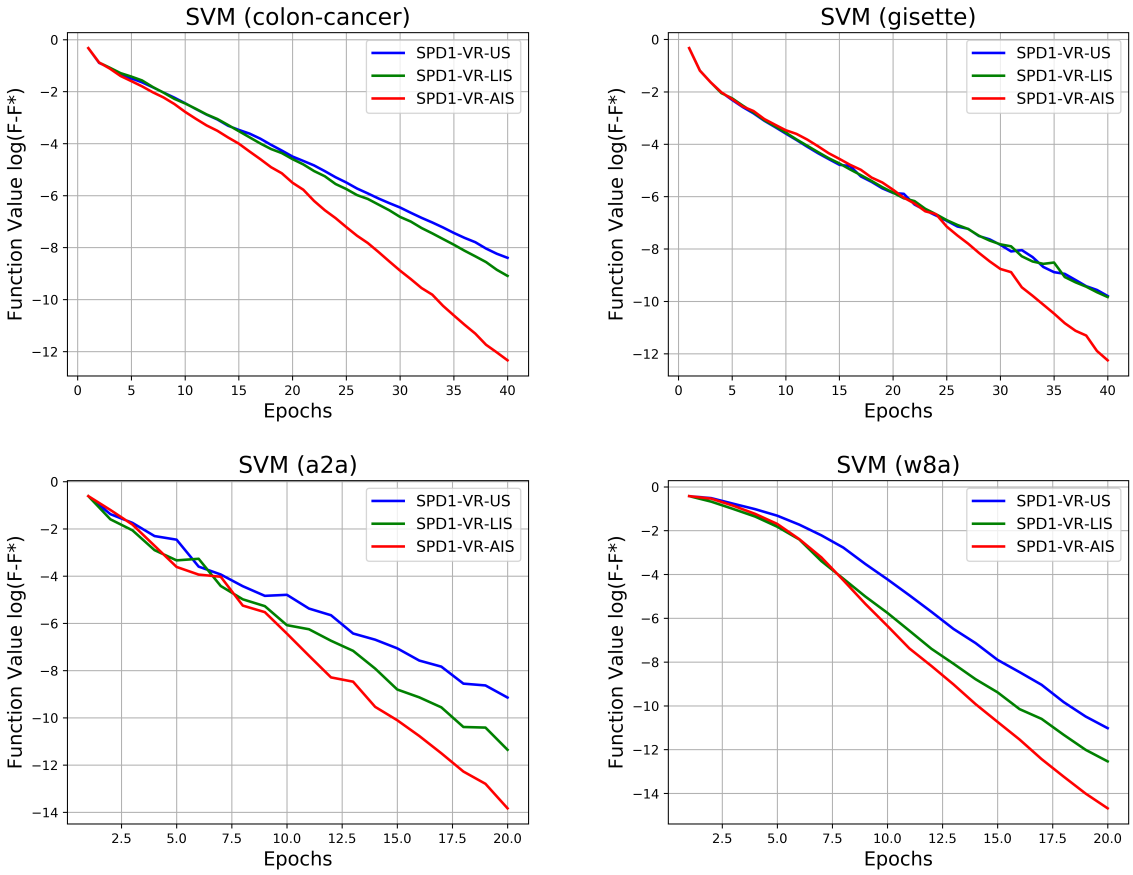


Fig. 6: Experimental Results for SPD1-based Algorithms

Table 5: Execution Time of DSPDC-based Algorithms

	colon-cancer	gisette	a2a	w8a
US	0.0150	2.05	0.0126	9.01
LIS	0.0185	2.21	0.0131	9.29
AIS	0.0195	2.30	0.0134	9.38

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Table 6: Execution Time of SPD1-VR-based Algorithms

	colon-cancer	gisette	a2a	w8a
US	0.0080	5.936	0.0235	2.621
LIS	0.0120	6.033	0.0270	3.0425
AIS	0.0135	6.399	0.0295	3.1155

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